ADDENDUM TO THE PAPER
SPECTRAL MULTIPLIER THEOREM FOR H^1 SPACES ASSOCIATED WITH SCHRÖDINGER OPERATORS WITH POTENTIALS SATISFYING A REVERSE HÖLDER INEQUALITY, ILLINOIS J. MATH 45 (2001), 1301–1313.

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ABSTRACT. The aim of this addendum is to explain actions of multiplier operators on H^1 spaces associated with Schrödinger operators with potentials satisfying a reverse Hölder inequality. In particular we show that boundedness of the operators $F(A)$ on atoms proved in [3] imply existence of their continuous extensions on H^1_A.

1. INTRODUCTION

Let $T_t(x,y)$ be the integral kernels a semigroup of linear operators $\{T_t\}_{t>0}$ on $\mathbb{R}^d$, $d \geq 3$, generated by a Schrödinger operator $-A = \Delta - V(x)$, where $V(x)$ is a non-zero non-negative function satisfying the reverse Hölder inequality with an exponent $q > d/2$, that is,

$$\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)^q\,dy\Big)^{1/q} \leq \frac{C}{|B(x,r)|}\int_{B(x,r)} V(y)\,dy$$

holds for every $x \in \mathbb{R}^d$ and $r > 0$. We say that an $L^1(\mathbb{R}^d)$-function f belongs to the Hardy space H^1_A if the maximal operator $\mathcal{M}_A f(x) = \sup_{t>0}|T_t f(x)|$ belongs to $L^1(\mathbb{R}^d)$. Then we set

$$\|f\|_{H^1_A} = \|\mathcal{M}_A f\|_{L^1(\mathbb{R}^d)}. \tag{1.1}$$

It was proved in [2] that the space H^1_A admits a special atomic decomposition, that is, every element $f \in H^1_A$ can be written as

$$f = \sum_{j=1}^{\infty} c_j a_j, \quad \sum_j |c_j| \leq C\|f\|_{H^1_A}, \tag{1.2}$$

where $c_j \in \mathbb{C}$ and a_j are special $(1,\infty)$-atoms for the space H^1_A. The following properties of the atoms will used in this addendum:

$$\text{every atom } a \text{ is supported by a ball } B \text{ and } \|a\|_{L^\infty} \leq |B|^{-1}.$$

Therefore $\|a\|_{L^1} \leq 1$, $\|a\|_{L^2} \leq |B|^{-1/2}$ and the convergence in (1.2) is in $L^1(\mathbb{R}^d)$. It is also shown in [2] that there is a constant $C > 0$ such that for every atom a one has

$$\|\mathcal{M}_A a\|_{L^1} \leq C.$$

2. Action of multiplier operators on H^1_A

Let $\int_0^\infty \lambda\, dE_A(\lambda)$ be the spectral resolution for A. For a Borel and bounded function F on $(0,\infty)$ we denote $F(A)=\int_0^\infty F(\lambda)\, dE_A(\lambda)$. It was actually proved in [3] that if a bounded continuous function F defined on $(0,\infty)$ satisfies

$$\sup_{t>0} \|\psi(\,\cdot\,)F(t\,\cdot\,)\|_{C(\alpha)} = C_0 < \infty \tag{2.1}$$

for certain $\alpha > d/2$ and a fixed nonzero auxiliary function $\psi \in C_c^\infty(0,\infty)$, then

$$\|F(A)a\|_{H^1_A} \leq CC_0 \text{ for every atom } a. \tag{2.2}$$

The aim of this addendum is to explain that the operator $F(A)$ has then the unique extension to continuous operator on H^1_A.

To do this let $f \in H^1_A \cap L^2(\mathbb{R}^d)$ and let $f=\sum_j c_j a_j$ be its atomic decomposition (see (1.2)). Clearly $F(A)f \in L^2(\mathbb{R}^d)$. Let g be a function of the form $g = T_s\varphi$ with $s>0$, and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Since $T_s(x,y)=T_s(y,x)$ satisfy the Gaussian bounds

$$0 \leq T_s(x,y) \leq Cs^{-d/2}\exp(-c|x-y|^2/s), \tag{2.3}$$

we get that $g \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. Therefore,

$$\int (F(A)f)(x)\overline{g(x)}\, dx = \int f(x)\overline{(\bar{F}(A)T_s\varphi)(x)}\, dx = \int f(x)\overline{(T_s\bar{F}(A)\varphi)(x)}\, dx. \tag{2.4}$$

Obviously, $F(A)\varphi \in L^2(\mathbb{R}^d)$. Hence, by (2.3), we have that $\overline{(T_s\bar{F}(A)\varphi)(x)} \in L^\infty(\mathbb{R}^d)$. Thus,

$$\begin{aligned}
\int (F(A)f)(x)\overline{g(x)}\, dx &= \int \sum_j c_j a_j(x)\overline{(T_s\bar{F}(A)\varphi)(x)}\, dx \\
&= \sum_j c_j \int a_j(x)\overline{(T_s\bar{F}(A)\varphi)(x)}\, dx \\
&= \sum_j c_j \int a_j(x)\overline{(\bar{F}(A)T_s\varphi)(x)}\, dx \\
&= \sum_j c_j \int (F(A)a_j)(x)\overline{(T_s\varphi)(x)}\, dx \\
&= \sum_j c_j \int (F(A)a_j)(x)\overline{g(x)}\, dx.
\end{aligned} \tag{2.5}$$

Assume now that $\varphi \in \mathcal{S}$. Recall that T_t is a strongly continuous semigroup of contractions on L^p for $1 \leq p < \infty$. Since $F(A)f \in L^2$, using (2.5), we obtain that

$$\begin{aligned}
\int (F(A)f)(x)\overline{\varphi(x)}\,dx &= \lim_{s\to 0} \int T_s F(A)f(x)\overline{\varphi(x)}\,dx \\
&= \lim_{s\to 0} \int F(A)f(x)\overline{T_s\varphi(x)}\,dx \\
&= \lim_{s\to 0} \sum_j c_j \int F(A)a_j(x)\overline{T_s\varphi(x)}\,dx \\
&= \lim_{s\to 0} \sum_j c_j \int (T_s F(A)a_j)(x)\overline{\varphi(x)}\,dx.
\end{aligned} \tag{2.6}$$

Observe that (2.2) implies that $\|T_s F(A)a_j\|_{L^1} \leq \|T_s F(A)a_j\|_{H^1_A} \leq CC_0$ uniformly on $s > 0$ and j, because the functions $\lambda \mapsto e^{-s\lambda}F(\lambda)$ satisfies (2.1) with a constant C_0 independent of $s > 0$. Therefore, we are allowed to change the order of the limit and the summation in (2.6) and obtain

$$\int (F(A)f)(x)\overline{\varphi(x)}\,dx = \sum_j c_j \int F(A)a_j(x)\overline{\varphi(x)}\,dx = \int \Big(\sum_j c_j F(A)a_j(x)\Big)\overline{\varphi(x)}\,dx. \tag{2.7}$$

So we get that for $f \in H^1_L \cap L^2(\mathbb{R}^d)$ one has:

$$F(A)f = \sum_j c_j F(A)a_j, \tag{2.8}$$

where $f = \sum_j c_j a_j$ is an atomic decomposition of f. Moreover,

$$\|F(A)f\|_{H^1_A} \leq CC_0 \|f\|_{H^1_A}.$$

The converges in (2.8) is in $L^1(\mathbb{R}^d)$ and also in H^1_A. If now $f_n \in L^2(\mathbb{R}^d)$ is a Cauchy sequence in H^1_A, then $F(A)f_n$ belong to $L^2(\mathbb{R}^d)$ and form a Cauchy sequence in H^1_A and of course in $L^1(\mathbb{R}^d)$.

References

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